

Generalized solutions of the Dirac equation, W bosons, and beta decay

Andrzej Okniński*

Chair of Mathematics and Physics, Politechnika Świętokrzyska,
Al. 1000-lecia PP 7, 25-314 Kielce, Poland

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Abstract

We study the 7×7 Hagen-Hurley equations describing spin 1 particles. We split these equations, in the interacting case, into two Dirac equations with non-standard solutions. It is argued that these solutions describe decay of a virtual W boson in beta decay.

1 Introduction

Recently, we have shown that in the free case covariant solutions of the $s = 0$ and $s = 1$ Duffin-Kemmer-Petiau (DKP) equations are generalized solutions of the Dirac equation [1]. These wavefunctions are non-standard since they involve higher-order spinors. We have demonstrated recently that in the $s = 0$ case the generalized solutions describe decay of a pion [2]. The aim of this work is to interpret spin 1 solutions, possibly in the context of weakly decaying particles.

There are several relativistic equations describing spin 1 particles, see [3, 4] for the reviews. The most common approach to study properties of spin 1 bosons is based on the 10×10 DKP equations (the DKP particles are bosons [5]). Several classes of potentials were used in DKP equations to investigate interactions of spin 1 particles [6–15]. However, we shall apply the 7×7 Hagen-Hurley equations [16–18] in spinor form [1, 19, 20]. Our motivation stems from the observation that these equations violate parity and thus should describe weakly interacting particles.

In the next Section we transform the Hagen-Hurley equations, in the interacting case, into two Dirac equations with non-standard solutions involving higher-order spinors, extending our earlier results described in [1]. These generalized solutions bear some analogy to generalized solutions of the Dirac equation argued to describe a lepton and three quarks [21]. In Section 3 we describe transition from non-standard solutions of two Dirac equations to the Dirac equation

*Email: fizao@tu.kielce.pl

for a lepton and the Weyl equation for a neutrino. In the last Section we show that the transition is consistent with decay of a virtual W boson in beta decay. In what follows we are using definitions and conventions of Ref. [22].

2 Generalized solutions of the Dirac equation in the interacting case

We have shown recently that, in the non-interacting case, solutions of the $s = 0$ and $s = 1$ DKP equations are generalized solutions of the Dirac equation [1]. In our derivation we have splitted the 10×10 DKP equations for $s = 1$ into two 7×7 Hagen-Hurley equations [16–18]. Let us note here that in the case of interaction with external fields such splitting is not possible since the identities (27) of Ref. [23], enabling the splitting, are not valid in the interacting case. Therefore, we shall base our theory on the 7×7 formulation, see Eqs. (18), (19) in [1] and Subsection 6 ii) in [19]. These equations violate parity P , where $P : x^0 \rightarrow x^0, x^i \rightarrow -x^i$ ($i = 1, 2, 3$), and thus one should expect a link with weak interactions.

We write one of these 7×7 equations (Eq. (19) of Ref. [1]), in the interacting case, in form:

$$\left. \begin{aligned} \pi^A_{\dot{B}} \zeta_{A\dot{D}} &= m\chi_{\dot{B}\dot{D}} \\ \pi_A^{\dot{D}} \chi_{\dot{B}\dot{D}} &= -m\zeta_{A\dot{B}} \end{aligned} \right\} \quad (1)$$

and it is assumed that

$$\chi_{\dot{B}\dot{D}} = \chi_{\dot{D}\dot{B}} \quad (2)$$

what is the $s = 1$ constraint. In Eqs. (1) we have $\pi^{A\dot{B}} = (\sigma^0 \pi^0 + \vec{\sigma} \cdot \vec{\pi})^{A\dot{B}}$, $\pi^\mu = p^\mu - qA^\mu$, σ^k ($k = 1, 2, 3$) are the Pauli matrices, and σ^0 is the 2×2 unit matrix. Let us note that equations (1), (2), which can be written in the 7×7 Hagen-Hurley form, were first proposed by Dirac [20].

Equations (1) in explicit form read:

$$\left. \begin{aligned} -(\pi^1 + i\pi^2)\chi_{1\dot{1}} - (\pi^0 - \pi^3)\chi_{2\dot{1}} &= -m\zeta_{1\dot{1}} \\ (\pi^0 + \pi^3)\chi_{1\dot{1}} + (\pi^1 - i\pi^2)\chi_{2\dot{1}} &= -m\zeta_{2\dot{1}} \\ -(\pi^1 - i\pi^2)\zeta_{1\dot{1}} - (\pi^0 - \pi^3)\zeta_{2\dot{1}} &= m\chi_{1\dot{1}} \\ (\pi^0 + \pi^3)\zeta_{1\dot{1}} + (\pi^1 + i\pi^2)\zeta_{2\dot{1}} &= m\chi_{2\dot{1}} \end{aligned} \right\} \quad (3a)$$

$$\left. \begin{aligned} -(\pi^1 + i\pi^2)\chi_{1\dot{2}} - (\pi^0 - \pi^3)\chi_{2\dot{2}} &= -m\zeta_{1\dot{2}} \\ (\pi^0 + \pi^3)\chi_{1\dot{2}} + (\pi^1 - i\pi^2)\chi_{2\dot{2}} &= -m\zeta_{2\dot{2}} \\ -(\pi^1 - i\pi^2)\zeta_{1\dot{2}} - (\pi^0 - \pi^3)\zeta_{2\dot{2}} &= m\chi_{1\dot{2}} \\ (\pi^0 + \pi^3)\zeta_{1\dot{2}} + (\pi^1 + i\pi^2)\zeta_{2\dot{2}} &= m\chi_{2\dot{2}} \end{aligned} \right\} \quad (3b)$$

where the condition $\chi_{\dot{B}\dot{D}} = \chi_{\dot{D}\dot{B}}$ is not imposed. We thus get two Dirac equations or, alternatively, a single Dirac equation with generalized solution

$$\mathbb{B} = \begin{pmatrix} \zeta_{1i} & \zeta_{1\dot{2}} \\ \zeta_{2i} & \zeta_{2\dot{2}} \\ \chi_{1i} & \chi_{1\dot{2}} \\ \chi_{2i} & \chi_{2\dot{2}} \end{pmatrix}$$

$$(\pi^0 \gamma^0 - \pi^1 \gamma^1 - \pi^2 \gamma^2 - \pi^3 \gamma^3) \mathbb{B} = m \mathbb{B}, \quad (4)$$

generalizing Eq. (24) of Ref. [1].

3 Decay of spin 1 bosons

We note that solutions of two Dirac equations (3) are non-standard since they involve higher-order spinors rather than spinors $\xi_A, \eta_{\dot{B}}$. To interpret Eqs. (3) we put:

$$\chi_{\dot{B}\dot{D}}(x) = \eta_{\dot{B}}(x) \alpha_{\dot{D}}(x) \quad (5a)$$

$$\zeta_{A\dot{B}}(x) = \xi_A(x) \alpha_{\dot{B}}(x) \quad (5b)$$

where $\alpha_{\dot{A}}(x)$ is the Weyl spinor while $\eta_{\dot{B}}(x), \xi_A(x)$ are the Dirac spinors. Note that now $\chi_{1\dot{2}} \neq \chi_{2\dot{1}}$ and, accordingly, the spin is not determined – more exactly, the spin is in the $0 \oplus 1$ space. It means that we consider virtual (off-shell) bosons. This substitution is in the spirit of the method of fusion of de Broglie [24, 25] (similar ansatz was used in the $s = 0$ case [2]). After the substitution of (5) into Eqs. (3) we obtain two equations:

$$\left. \begin{aligned} -(\pi^1 + i\pi^2) \eta_1 \alpha_{\dot{A}} - (\pi^0 - \pi^3) \eta_2 \alpha_{\dot{A}} &= -m \xi_1 \alpha_{\dot{A}} \\ (\pi^0 + \pi^3) \eta_1 \alpha_{\dot{A}} + (\pi^1 - i\pi^2) \eta_2 \alpha_{\dot{A}} &= -m \xi_2 \alpha_{\dot{A}} \\ -(\pi^1 - i\pi^2) \xi_1 \alpha_{\dot{A}} - (\pi^0 - \pi^3) \xi_2 \alpha_{\dot{A}} &= m \eta_1 \alpha_{\dot{A}} \\ (\pi^0 + \pi^3) \xi_1 \alpha_{\dot{A}} + (\pi^1 + i\pi^2) \xi_2 \alpha_{\dot{A}} &= m \eta_2 \alpha_{\dot{A}} \end{aligned} \right\} \quad (6)$$

where $\dot{A} = \dot{1}, \dot{2}$, and, after substituting solution of the Weyl equation

$$p^{A\dot{B}} \alpha_{\dot{B}} = 0, \quad (7)$$

$\alpha_{\dot{A}}(x) = \hat{\alpha}_{\dot{A}} e^{ik \cdot x}$, $k^\mu k_\mu = 0$, we get a single Dirac – equation for spinors $\xi_A(x), \eta_{\dot{B}}(x)$:

$$\left. \begin{aligned} -(\tilde{\pi}^1 + i\tilde{\pi}^2) \eta_1 - (\tilde{\pi}^0 - \tilde{\pi}^3) \eta_2 &= -m \xi_1 \\ (\tilde{\pi}^0 + \tilde{\pi}^3) \eta_1 + (\tilde{\pi}^1 - i\tilde{\pi}^2) \eta_2 &= -m \xi_2 \\ -(\tilde{\pi}^1 - i\tilde{\pi}^2) \xi_1 - (\tilde{\pi}^0 - \tilde{\pi}^3) \xi_2 &= m \eta_1 \\ (\tilde{\pi}^0 + \tilde{\pi}^3) \xi_1 + (\tilde{\pi}^1 + i\tilde{\pi}^2) \xi_2 &= m \eta_2 \end{aligned} \right\} \quad (8)$$

where $\tilde{\pi}^\mu \equiv \pi^\mu + k^\mu$, since components $\alpha_{\dot{1}}(x), \alpha_{\dot{2}}(x)$ cancel out.

Equations (7), (8) describe two spin $\frac{1}{2}$ particles, whose spins can couple to $s = 0$ or $s = 1$, i.e. $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$.

4 Conclusions

Results obtained in Sections 2, 3 cast new light on the Hagen-Hurley equations as well as on weak decays of spin 1 bosons. We have shown that transition from equation (1), describing a spin $s = 1$ particle, to equations (7), (8), via substitution (5) – which means that now $s \in 0 \oplus 1$, corresponds to decay of this particle into a Weyl antineutrino, cf. Eq. (7), and a Dirac lepton, cf. Eq. (8). Indeed, it should be a weak decay since Eq. (1) violates parity. The spin of this particle becomes undetermined in the process of decay, more exactly it belongs to the $0 \oplus 1$ space – this suggests that this is a virtual particle. Therefore, the products, a lepton and a antineutrino, should have total spin 0 or 1 and there should be a third particle to secure spin conservation.

The above description fits a (three-body) beta decay with formation of a virtual W^- boson, decaying into a lepton and antineutrino. This is most conveniently explained in the case of a mixed beta decay [26]:

$$n(\uparrow) \longrightarrow \begin{cases} p(\downarrow) + [e(\uparrow) \bar{\nu}_e(\uparrow)] & \text{Gamow-Teller transition} \\ p(\uparrow) + [e(\uparrow) \bar{\nu}_e(\downarrow)] & \text{Fermi transition} \end{cases} \quad (9)$$

where products of the W^- boson decay (see [27]) are shown in square brackets and (\uparrow) denotes spin $\frac{1}{2}$ – this seems to correspond well to the proposed transition from Eq. (1) to Eqs. (7), (8). Since spin of the products of decay of the virtual W^- boson belongs to the $0 \oplus 1$ space, their spin can be $s = 0$ or $s = 1$. Moreover, in the case of the Gamow-Teller transition there must be a spin-flip in the decaying nucleon. Let us add here, that in the reaction (9) some neutrons (82%) decay according to the Gamow-Teller mechanism while some (18%) undergo the Fermi transition [26]. This mixed mechanism is explained by decoupled spins of the just born products – indeed, the condition $\chi_{1\dot{2}} = \chi_{\dot{2}1}$ for the spinor $\chi_{\dot{A}\dot{B}}$, due to the substitution (5a), does not hold and spin of the products is in the $0 \oplus 1$ space.

It is now obvious that another set of 7×7 equations, involving spinor η_{AB} rather than $\chi_{\dot{A}\dot{B}}$, see Eq. (18) of Ref. [1], describes a β^+ decay with intermediate W^+ boson. Let us note finally, that kinematics of the neutrino appears in the Dirac equation for the electron with arbitrary neutrino four-momentum, suggesting a continuous distribution of neutrino energy.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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